交通工程学
Introduction to Traffic Engineering
第 11 节　线性规划简介[1]

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[1]基于美国亚利桑那州立大学 Angelia Nedich 教授讲义

本节目录

运筹学

- 对于工科、经济、金融等学科都至关重要的学科
- 提供了用于系统分析和决策支持的工具
  - 最优的分配/排班/资源管理
  - 不确定环境下长期的决策：网约车的调度?
  - 服务系统的最优生产策略
- 各行各业每天都会对这些“小”问题进行决策
- 运筹学能做什么?
  - 提供了一套用于分析该类问题的共通的框架
  - 使我们有能力开发用于解决该类问题的理论方法
  - 使我们有能力开发用于求解大型问题的算法和软件

运筹学的内容

- 以解决复杂决策问题为目标的数学模型和工具
- 一开始用于军事后勤领域（二战时的英美），逐渐扩展到制造、交通、物流、金融、医疗等领域
- 包含以下内容：
  - 数学优化：线性/非线性/整数/动态/凸/非凸/半正定/半无限/锥/随机/Robust 优化[2]
  - 排队论
  - 网络模型
  - 决策分析
  - 博弈论
  - 数理金融
  - 仿真与求解软件等
- 时间所限，我们仅涉及线性规划、整数规划和网络模型的初步知识

[2] 以上内容有交集

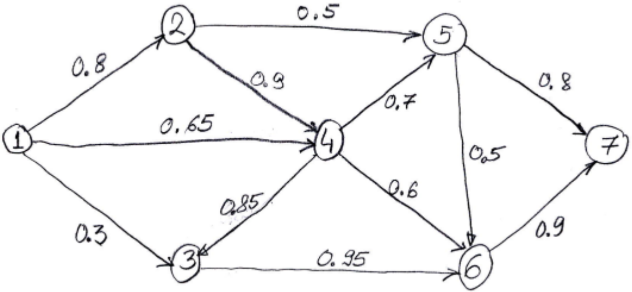
例子：资源分配

- 某公司生产两种类型的雪糕 A，B
  - 两种雪糕每公斤的利润分别是 50 和 60 人民币
  - 为了生产冰淇淋，需要两种半成品
  - 半成品 1：每公斤雪糕 A 需要 600g，雪糕 B 需要 500g
  - 半成品 2：每公斤雪糕 A 需要 400g，雪糕 B 需要 500g
  - 两种半成品的采购量分别为 20 吨和 10 吨
- 应该分别生产多少雪糕 A 和 B，以获得最大的企业效益?

例子：投资

- A Trader wants to invest a sum of money that would generate an annual yield of at least \$10,000
- Three stock groups are available: blue chips, com and high tech, with average annual yields of 10%, 25% and 35%,respectively
- Though com and high-tech stocks provide higher yield, they are more risky, and the Trader wants to limit the amount invested in these stocks to no more than 40% and 30%, resp., of the total investment
- Problem: What is the minimum amount the Trader should invest in each stock group to accomplish the investment goal?

例子: 网络可靠性



- Consider the communication network between stations 1 and 7
  - Messages are sent from station 1 to station 7.
  - Each link in the network may or may not operate.
  - The probability that a link will operate is shown on each arc.
  - Problem: Find the most reliable route, i.e., a route that will maximize the probability of a successful transmission.

问题定义

- 决策的因素 → 决策变量
- 决策的目标 → 目标函数
- 决策所受的限制 → 约束条件

雪糕厂例子回顾

- 决策变量：x_1, x_2 分别为两种雪糕每天生产的公斤数
- 目标函数：利润最大化：$\max 50x_1 + 60x_2$
- 约束条件：材料量的限制：$0.6x_1 + 0.4x_2 \leqslant 20,000$，$0.5x_1 + 0.5x_2 \leqslant 10,000$，$x_1 \geqslant 0$，$x_2 \geqslant 0$

线性规划模型

$$\max \qquad z = 50x_1 + 60x_2$$

subject to

$$0.6x_1 + 0.4x_2 \leqslant 20,000$$
$$0.5x_1 + 0.5x_2 \leqslant 10,000$$
$$x_1,\ x_2 \geqslant 0$$

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线性规划模型认知

- 目标函数为线性函数

$$z = a_1x_1 + \cdots + a_nx_n$$

  其中 a_i 为给定的参数
- 所有约束条件均为线性等式或不等式

$$c_1x_1 + \cdots + c_nx_n \leqslant b$$
$$c_1x_1 + \cdots + c_nx_n \geqslant b$$
$$c_1x_1 + \cdots + c_nx_n = b$$

  其中 c_i, b 为给定的参数

其他变种

$$\begin{aligned} \min \quad & z = 4x_1^2 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geqslant 2 \\ & 2x_1 - x_2 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \min \quad & z = 4x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 x_2 \geqslant 2 \\ & 2x_1 - x_2 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \min \quad & z = 4x_1 + 5x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geqslant 2 \\ & 2x_1 - x_2 \geqslant 0 \\ & x_1, x_2 \text{ are integers} \\ & \text{i.e., } x_1, x_2 \in \{0, 1, 2, \dots\} \end{aligned}$$

如何求解?

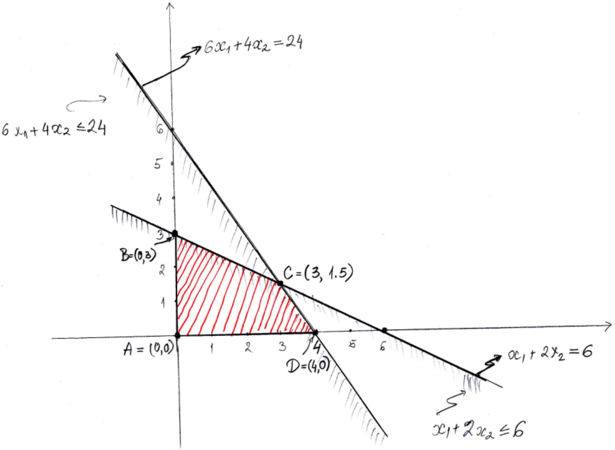
- 图解法
  - 把每个约束条件画出来
  - 圈定取值的区域（可行域）
  - 画出目标函数的等值曲线
  - 朝着目标函数等值曲线的梯度方向移动（最大化问题，最小化问题反之）

图解法示例

问题

$$\max z = 5x_1 + 4x_2$$

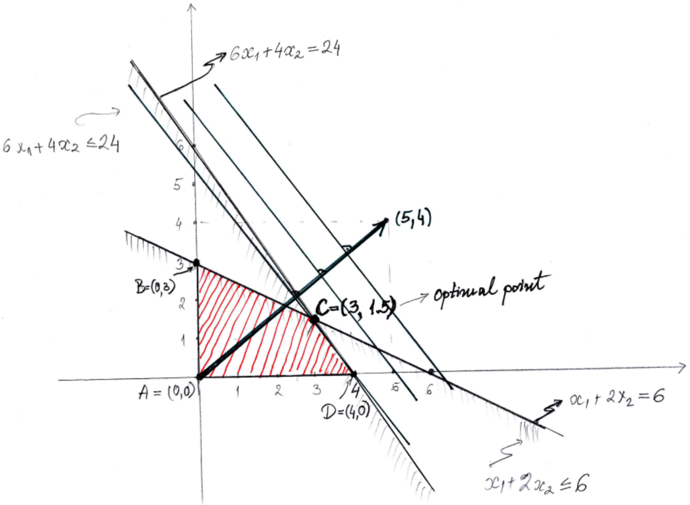
$$\text{s.t. } 6x_1 + 4x_2 \leqslant 24$$
$$x_1 + 2x_2 \leqslant 6$$
$$x_1, 2x_2 \geqslant 0$$



图解法示例

- The constraint set is a polyhedral set (intersection of finitely many half-spaces)
- The intersection of any of two lines (corresponding to two different constraints) defines a **corner** or a **vertex**（顶点）
  - In our example, the vertices are A = (0, 0), B = (0, 3), C = (3, 1.5), D = (4, 0), (0, 6), and (6, 0)
- Some vertices are feasible for the problem, and some are not
  - In our example, A, B, C, and D are feasible (belong to the constraint set), while (0, 6), and (6, 0) are not feasible (lie outside of the constraint set)

Objective $z = 5x_1 + 4x_2$ to be maximized



Figure

图解法的局限

- 一般图示法仅能求解两个变量的问题
- 当变量数目多于 2 时，一种通用的线性规划求解方法是**单纯形法**。单纯形法的灵感来自顶点在图解法的作用

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为什么需要标准型?

- 单纯形法一般被视为实际使用时最有效率的 LP 求解方法[3]，也是商业求解器（cplex, gurobi, 国产的 copt 等）和开源工具（coinor-clp, lpsolve 等）中通用的方法
- 单纯形法的实施，需要把 LP 问题转化成标准型

[3]目前尚无理论论验证其的确是效率最高的。事实上，在极端情况下，需要访问每一个顶点。因此，一般是用椭球法 (ellipsoid method) 或内点法 (interior point method) 证明线性规划能在多项式时间求解

标准型的形式

- 即约束条件满足特定形式的 LP 模型

$$
\begin{aligned}
\max(\min) z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to } & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
& x_1 \geqslant 0, x_2 \geqslant 0, \ldots, x_n \geqslant 0
\end{aligned}
$$

- m 个等式约束，n 个非负变量，且 $m \leqslant n$

转换方法

- $\geqslant$ 符号的约束：添加一个剩余变量（surplus variable）
- $\leqslant$ 符号的约束：添加一个松弛变量（slack variable）
- 不限制符号的变量：转换成两个新变量的差
- 所有新引入的变量均为非负，松弛变量和剩余变量在目标函数中的成本取值应为 0

示例

原型

$$\begin{aligned} \min \quad & z = 3x_1 + 8x_2 + 4x_3 \\ \text{s.t.} \quad & x_1 + x_2 \geqslant 8 \\ & 2x_1 - 3x_2 \leqslant 0 \\ & x_2 \geqslant 9 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

标准型

$$\begin{aligned} \min \quad & z = 3x_1 + 8x_2 + 4x_7 - 4x_8 \\ \text{s.t.} \quad & x_1 + x_2 - x_4 = 8 \\ & 2x_1 - 3x_2 + x_5 = 0 \\ & x_2 - x_6 = 9 \\ & x_1, x_2, x_4, x_5, x_6, x_7, x_8 \geqslant 0 \end{aligned}$$

x_3 已经被 x_7, x_8 代替，因此不再存在于本问题中。这时，$m = 3, n = 7$

本节目录

基[4]解

- For a problem in the standard form a basic solution is a point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ that has at least $n - m$ coordinates equal to 0, and satisfies all the equality constraints of the problem

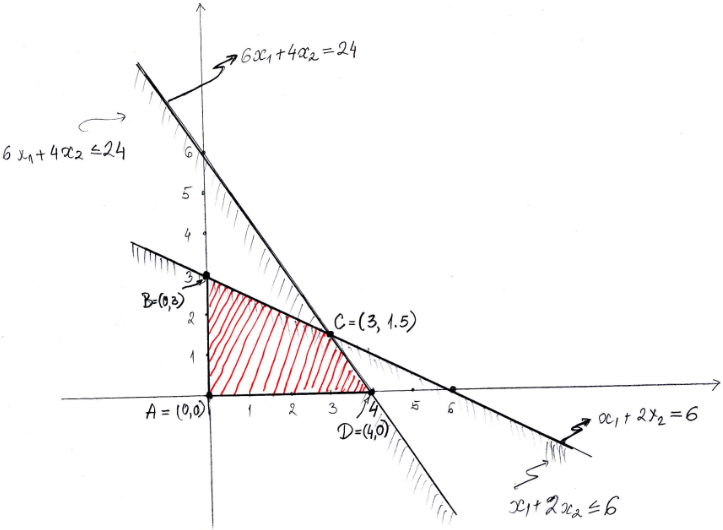
$$
\begin{aligned}
\max(\min) z &= c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n \\
\text{subject to } & a_{11}\bar{x}_1 + a_{12}\bar{x}_2 + \cdots + a_{1n}\bar{x}_n = b_1 \\
& a_{21}\bar{x}_1 + a_{22}\bar{x}_2 + \cdots + a_{2n}\bar{x}_n = b_2 \\
& \vdots \\
& a_{m1}\bar{x}_1 + a_{m2}\bar{x}_2 + \cdots + a_{mn}\bar{x}_n = b_m
\end{aligned}
$$

- If the point $\bar{x}$ has all components nonnegative, i.e., $\bar{x}_i \geqslant 0$ for all i, then $\bar{x}$ is a basic feasible solution （基可行解）
- Otherwise, (i.e., if $\bar{x}_j < 0$ for some index j), $\bar{x}$ is basic infeasible solution （基不可行解）

[4] 此处“基”指代初始的状态

基解

◇ 基解位于顶点处



如何用代数方法获得基解

- If the problem is not in standard form, bring it to the standard form
- Basic solutions are determined from the standard form as follows:
  - Select $n - m$ out of n nonnegative inequalities (coordinate indices) i, $x_i \geqslant 0, i = 1, \ldots, m$ and set them to zero
    $x_j = 0$ for a total of $n - m$ indices j (nonbasic variables 非基变量)
  - Substitute these zero values in the equalities: we have m unknown variables an m equalities
  - Solve this $m \times m$ system of equations: we obtain values for the remaining m variables (basic variables 基变量)
  - If these m (basic) variables are nonnegative, we have a basic feasible solution; otherwise, a basic infeasible solution

示例

原型

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} & 6x_1 + 4x_2 \leqslant 24 \\ & x_1 + 2x_2 \leqslant 6 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} & 6x_1 + 4x_2 + x_3 = 24 \\ & x_1 + 2x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{aligned}$$

- We have $m = 2$ and $n = 4$. Thus, when determining the basic solutions, we set 2 indices to zero
- Suppose we choose indices 1, 2 and set $x_1 = 0$ and $x_2 = 0$
- Substituting these in the equations yields: $x_3 = 24$ and $x_4 = 6$
- Corresponding basic solution is $x = (0, 0, 24, 6)$ and it is a basic feasible solution
- In this solution, x_1 and x_2 are nonbasic variables, while x_3 and x_4 are basic variables

用穷尽搜索的方法寻找最优解

- 通过凸优化理论，我们可知（当存在最优解时,）最优解一定位于某一顶点
- 因此可用一下遍历算法求解线性规划问题

基变量	基解	是否位于可行域	目标函数值
x_1, x_2	3, 1.5	是	21
x_1, x_3	6, -12	否	NA
x_1, x_4	4, 0	是	20
x_2, x_3	3, 12	是	15
x_2, x_4	6, -6	否	NA
x_3, x_4	24, 6	是	0

- Thus, the optimal solution is $x_1 = 3, x_2 = 1.5, x_3 = 0$, and $x_4 = 0$ and the optimal value is $z = 21$.
- In this case, we have only one solution
- Computing the objective value for each feasible solution
- 课下可以尝试将每个基解与图解法中的顶点对应

为何穷尽搜索不是一个好方法

对于大规模问题，该方法无法使用

- 盲目搜索非常无效
- Given an LP in the standard form with m equations and n variables, the number of basic solutions is

$$\frac{n(n-1)(n-2)\dots(n-m+1)}{m!}$$

- Say m = 4 and n = 8, then there are 70 solutions
- It is hard to "manually" list them all and find the best
- We will use a more efficient method (simplex method) to perform a "smarter" search (selectively moving to a better point)

为何穷尽搜索不是一个好方法

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示例

原型

$$\max \quad z = 5x_1 + 4x_2$$
$$\text{s.t.} 6x_1 + 4x_2 \leqslant 24$$
$$x_1 + 2x_2 \leqslant 6$$
$$x_1, x_2 \geqslant 0$$

标准形

$$\max \quad z = 5x_1 + 4x_2$$
$$\text{s.t.} 6x_1 + 4x_2 + x_3 = 24$$
$$x_1 + 2x_2 + x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geqslant 0$$

- The basic variables are also referred to as a basis. 基变量一般也被成为基
- Every basis has exactly m variables.
- Substituting these in the equations yields: $x_3 = 24$ and $x_4 = 6$

单纯形法[5]的步骤——初始化

初始化

1. Choose an initial basic feasible solution
2. Check if it is optimal (if yes, we are done)
   - Suppose we choose x_3 and x_4 as basis
   - We solve the equations in terms of x_1 and x_2 to find the basic feasible solution corresponding to this basis
   - When $x_1 = x_2 = 0$, we have the basic feasible solution "readily" available $x_1 = 0, x_2 = 0, x_3 = 24, x_4 = 6$

Basis	Equations	RHS Values
(x_3)	$6x_1 + 4x_2 + x_3$	$= 24$
(x_4)	$x_1 + 2x_2 \quad + x_4$	$= 6$

Item 2: Is this optimal?

[5]由于时间所限，我们仅涉略单纯形法的算法，而不深析其原理，如对原理感兴趣，课下可阅读 Bertsimas &Tsitsiklis 合著的 Introduction to Linear Optimization

单纯形法的步骤-最优性检验

检验当前解是否最优

- Express the cost in terms of nonbasic variables

$$z = 5x_1 + 4x_2$$

- In the basic solution (0, 0, 24, 6) the nonbasic variables are zero.
- Would increasing any of these values increase the objective value (max)?
- Yes, actually increasing either x_1 or x_2 would improve the cost.(试思考：为何允许增大 x_1 或 x_2 的取值？)
- So we can choose any of these variables, say we select x_1.

单纯形法的步骤——最优性检验 (cont.)

一种更简单的检验最优性的方法

Basis	Equations	RHS Values
$(z)-row$	$z-5x_1-4x_2$	$=0$
(x_3)	$6x_1+4x_2+x_3$	$=24$
(x_4)	$x_1+2x_2 \quad +x_4$	$=6$

- There are **negative coefficients** in $z-$row (associated with x_1 and x_2)
- **The variables with negative coefficients indicate directions of improvement for the objective value** (when **maximizing**)
- Current basic solution (0,0,24,6) is not optimal: x_1 or x_2 can be increased to improve the $z-$value
- Next step: move to a better basic feasible solution by taking either x_1 or x_2 in the basis
- Suppose we choose x_1

单纯形法的步骤——下一次迭代 (cont.)

- 更新基变量
- 更新基变量对应的解
- 检验最优性

具体如下:

- We have x_3 and x_4 as current basis and the nonoptimal solution (0, 0, 24, 6)
- We selected variable x_1 to enter the basis, but one of the current basic variables, x_3 or x_4, has to leave the basis
- x_1 entering the basis means that x_1 value is increasing from its current value 0
- Is there anything prohibiting us to increase x_1 as much as we want?
- The answer is the constraint equations

更新基变量

Basis	Equations	RHS Values
$(z) - row$	$z - 5x_1 - 4x_2$	$= 0$
(x_3)	$6x_1 + 4x_2 + x_3$	$= 24$
(x_4)	$x_1 + 2x_2 \quad + x_4$	$= 6$

$$\mathbf{6x_1} + 4x_2 + x_3 = \mathbf{24} \implies x_1 \text{ can be at most } 24/6 = 4$$

$$\mathbf{x_1} + 2x_2 + x_4 = \mathbf{6} \implies x_1 \text{ can be at most } 6/1 = 6$$

- The variable corresponding to the smaller ratio leaves the basis.
- Thus, x_3 leaves the basis. The new basis is x_1 and x_4.

更新对应于新的基变量的基可行解:In other words, we need to resolve the relations to express the new basic variables x_1 and x_4 in terms of nonbasic variables x_2 and x_3.

Basis	Equations	RHS Values
$(z)-row$	$z-5x_1-4x_2$	$=0$
(x_1) x_3 **left**	$\mathbf{6x_1}+4x_2+x_3$	$=24$
(x_4)	$x_1+2x_2 \quad +x_4$	$=6$

使用高斯消元法（Gauss-Jordan elimination）

Gauss-Jordan: Divide the equation by 6 to have coeff. of x_1 equal to 1

Basis	Equations	RHS Values
$(z)-row$	$z-5x_1-4x_2$	$=0$
(x_1)	$x_1+\frac{2}{3}x_2+\frac{1}{6}x_3$	$=4$
(x_4)	$x_1+2x_2 \quad +x_4$	$=6$

Gauss-Jordan: Eliminate x_1 from the second equation

Basis	Equations	RHS Values
$(z)-row$	$z-5x_1-4x_2$	$=0$
(x_1)	$x_1+\frac{2}{3}x_2+\frac{1}{6}x_3$	$=4$
(x_4)	$\frac{4}{3}x_2-\frac{1}{6}x_3+x_4$	$=2$

新的基可行解为 (4,0,0,2)

新解是否最优?

Gauss-Jordan: Eliminate x_1 from the z-row in the last table

Basis	Equations	RHS Values
$(z) - row$	$z \quad - \frac{2}{3}x_2 + \frac{5}{6}x_3$	$= 20$
(x_1)	$x_1 + \frac{2}{3}x_2 + \frac{1}{6}x_3$	$= 4$
(x_4)	$\frac{4}{3}x_2 - \frac{1}{6}x_3 + x_4$	$= 2$

- The current basic solution is not optimal! Why?
- Thus, we have to perform another iteration.
- Which of the currently nonbasic variables, x_2 or x_4, when increased will result in increased z-value?

单纯形法的步骤——下一次迭代

- 更新基变量
- 更新基变量对应的解
- 检验最优性

具体如下

- We have x_1 and x_4 as current basis
- We select variable x_2 to enter the basis, but one of the current basic variables, x_1 or x_4, has to leave the basis
- Which variable will leave the current basis? Perform ratio test

Basis	Equations	RHS Values
$(z) - row$	$z \quad -\frac{2}{3}x_2 + \frac{5}{6}x_3$	$= 20$
(x_1)	$x_1 + \frac{\mathbf{2}}{\mathbf{3}}x_2 + \frac{1}{6}x_3$	$= \mathbf{4}$
(x_4)	$\frac{\mathbf{4}}{\mathbf{3}}x_2 - \frac{1}{6}x_3 + x_4$	$= \mathbf{2}$

$$x_1 + \frac{\mathbf{2}}{\mathbf{3}}x_2 + \frac{1}{6}x_3 = \mathbf{4} \implies x_2 \text{ can be at most } \frac{4}{\frac{2}{3}} = 6$$

$$\frac{\mathbf{4}}{\mathbf{3}}x_2 - \frac{1}{6}x_3 + x_4 = \mathbf{2} \implies x_2 \text{ can be at most } \frac{2}{\frac{4}{3}} = \frac{6}{4}$$

- The variable corresponding to the smaller ratio leaves the basis.
- Thus, x_4 leaves the basis. The new basis is x_1 and x_2

更新基可行解

Basis	Equations	RHS Values
$(z)-row$	$z \quad -\frac{2}{3}x_2+\frac{5}{6}x_3$	$=20$
(x_1)	$x_1+\frac{2}{3}x_2+\frac{1}{6}x_3$	$=4$
(x_2) x_4 **left**	$\mathbf{\frac{4}{3}}x_2-\frac{1}{6}x_3+x_4$	$=2$

- The number $\frac{4}{3}$ is used to eliminate x_2 from x_2-row and $z-$row
- This number is referred as pivot element

高斯消元法得到

Basis	Equations	RHS Values
$(z)-row$	$z \quad + \frac{11}{12}x_3 + \frac{1}{2}x_4$	$= 21$
(x_1)	$x_1 \quad + \frac{1}{4}x_3 - \frac{1}{2}x_4$	$= 3$
(x_2)	$x_2 - \frac{1}{8}x_3 + \frac{3}{4}x_4$	$= \frac{3}{2}$

- What is the current basic solution (corresponding to the data above).
- Is this optimal? - Look at the coefficients of x_3 and x_4 in z-row.
- They are nonnegative - so the current basic solution is optimal
- What is the optimal objective value?

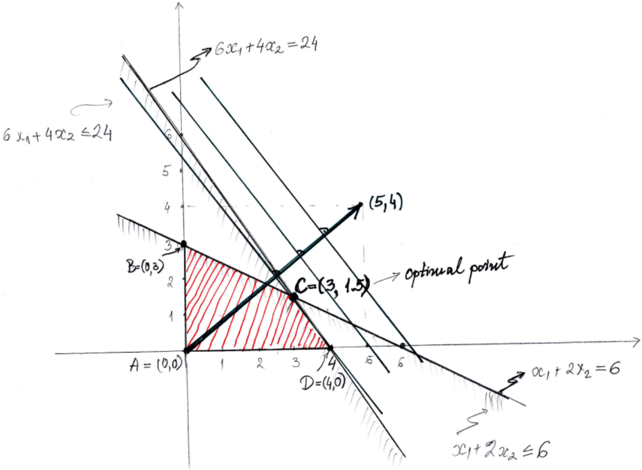

Figure: (0, 0) → (4, 0) → (3, 1.5)

更多关于单纯形法的内容

- 对于大规模的线性规划问题，如何找到初始解？ → 两阶段单纯形法
- 特殊的解
  - 退化：Degeneracy is a term used for a basic feasible solution having one or more basic variables at value 0
  - 多个解
  - 无界解
  - 无解

本节目录

敏感性分析

- Given a solution to an LP problem, one may ask how sensitive the solution is to the changes in the problem data
  - By how much can the RHS of the constraints change without causing changes in the current optimal basis?
  - By how much one or more coefficients in the objective cost may change without causing changes in the current optimal basis?
- The sensitivity (stability) of the solution provides the answers
- The sensitivity analysis provides us with ranges of possible changes in the problem data without causing changes in the optimal basis

敏感性分析——资源约束的变化

当一个或者多个约束条件的右手侧取值发生改变时，解的性质如何改变?

示例

TOYCO production of toys (trains, trucks, cars) that require processing on three machines

maximize	$z = 3x_1 + 2x_2 + 5x_3$	
subject to	$x_1 + 2x_2 + x_3 \leq 430$	available time on machine 1
	$3x_1 + 2x_3 \leq 460$	available time on machine 2
	$x_1 + 4x_2 \leq 420$	available time on machine 3
	$x_1, x_2, x_3 \geq 0$	

最优解

Introducing the slack variables $x_4, x_5, x_6 \geqslant 0$ and using them as initial basis, we find an optimal solution (table)

Basis	x_1	x_2	x_3	x_4	x_5	x_6	RHS Values
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Suppose the simultaneous changes occur in the availability of machines; the problem takes the form

$$
\begin{array}{lll}
\text{maximize} & z = 3x_1 + 2x_2 + 5x_3 & \\
\text{subject to} & x_1 + 2x_2 + x_3 \leq 430 + \mathbf{D_1} & \text{available time on machine 1} \\
 & 3x_1 \qquad + 2x_3 \leq 460 + \mathbf{D_2} & \text{available time on machine 2} \\
 & x_1 + 4x_2 \qquad \leq 420 + \mathbf{D_3} & \text{available time on machine 3} \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

What are the ranges of D_1, D_2, D_3 that would not change the solution?

Form an initial table and treat D_1, D_2, D_3 as part of the RHS solution

							Solution		**side**	
Basis	x_1	x_2	x_3	x_4	x_5	x_6	RHS Values	D_1	D_2	D_3
z	-3	-2	-5	0	0	0	0	0	0	0
x_4	1	2	1	1	0	0	430	1	0	0
x_5	3	0	2	0	1	0	460	0	1	0
x_6	1	4	0	0	0	1	420	0	0	1

Perform the simplex method - what will the columns D_i look like in the optimal table (at the end of the simplex method)?

Basis	x_1	x_2	x_3	x_4	x_5	x_6	Solution RHS Values	D_1	side D_2	D_3
z	4	0	0	1	2	0	$= 1350$	1	2	0
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	$=$ 100	$\frac{1}{2}$	$-\frac{1}{4}$	0
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	$=$ 230	0	$\frac{1}{2}$	0
x_6	2	0	0	-2	1	1	$=$ 20	-2	1	1

The basic variables and $z-$value relations: dependency on D_i

$$
\begin{aligned}
z &= 1350 + D_1 + 2D_2 \\
x_2 &= 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \\
x_3 &= 230 + \frac{1}{2}D_2 \\
x_6 &= 20 - 2D_1 + D_2 + D_3
\end{aligned}
$$

The ranges of D_i that do not cause the change in the optimal basis are the ranges for which the solution in the preceding table is feasible:

$$x_2 \geqslant 0 \Longrightarrow 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \geqslant 0$$
$$x_3 \geqslant 0 \Longrightarrow 230 + \frac{1}{2}D_2 \geqslant 0$$
$$x_6 \geqslant 0 \Longrightarrow 20 - 2D_1 + D_2 + D_3 \geqslant 0$$

- Solving this system of inequalities will give range of values for Di for which the current optimal basis (x_2, x_3, x_6) remains optimal.
- The preceding relations also can provide answers on feasibility of the current basis when only one or only two of the rhs values change i.e., set $D_i = 0$ if the right-hand side of the $i-$th constraint has not changed

敏感性分析：目标函数成本参数的变化

TOYCO production of toys (trains, trucks, cars) that require processing on three machines

$$
\begin{aligned}
\text{maximize} \quad & z = 3x_1 + 2x_2 + 5x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \le 430 \qquad \text{available time on machine 1} \\
& 3x_1 \qquad\quad + 2x_3 \le 460 \qquad \text{available time on machine 2} \\
& x_1 + 4x_2 \qquad\quad \le 420 \qquad \text{available time on machine 3} \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Suppose the simultaneous changes are to occur in the profits; the problem takes the form

$$\begin{aligned}
\text{maximize} \quad & z = (3 + d_1)x_1 + (2 + d_2)x_2 + (5 + d_3)x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \le 430 \\
& 3x_1 \qquad + 2x_3 \le 460 \\
& x_1 + 4x_2 \qquad \le 420 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

What are the ranges of d_1, d_2, d_3 that would not change the optimal solution?

Form an initial table and treat d1, d2, d3 as part of the cost

Basis	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	$-3-d_1$	$-2-d_2$	$-5-d_3$	0	0	0	0
x_4	1	2	1	1	0	0	430
x_5	3	0	2	0	1	0	460
x_6	1	4	0	0	0	1	420

Perform the simplex method - what will the "reduced costs" look like in the optimal table?

Basis	x_1	x_2	x_3	x_4	x_5	x_6	RHS Values
z	$4-\frac{d_2}{4}+\frac{3d_3}{2}-d_1$	0	0	$1+\frac{d_2}{2}$	$2-\frac{d_2}{4}+\frac{d_3}{2}$	0	$1350+100d_2+230d_3$
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

The optimal table of the original problem

Basis	x_1	x_2	x_3	x_4	x_5	x_6	RHS Values
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

- The ranges of d_i that do not cause the change in the optimal solution are the ranges for which the the preceding table stays optimal
- Since it is maximization, we need the reducd costs to remain nonnegative

$$
\begin{aligned}
&(x_1) \qquad 4 - \frac{d_2}{4} + \frac{3d_3}{2} - d_1 \geq 0 \\
&(x_4) \qquad 1 + \frac{d_2}{2} \geq 0 \\
&(x_5) \qquad 2 - \frac{d_2}{4} + \frac{d_3}{2} \geq 0
\end{aligned}
$$

The optimal value is changing according to the following rule:
$z = 1350 + 100d_2 + 230d_3$

自学内容

- 单纯形表
- 对偶理论
- 对偶单纯形法
- …

软件/求解器

- 商业求解器：gurobi, cplex, xpress
- 开源软件：coin-or clp, lpsolve
- 建模工具：python+pulp

谢谢！